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# The Maxwell–Bloch equation and correlation functions for the penetrable Bose gas

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**Abstract.** We consider the quantum nonlinear Schrödinger equation in one space and one time dimension. We are interested in the non-free-fermionic case. We consider static temperature-dependent correlation functions. The determinant representation for the correlation functions simplifies in the small-mass limit of the Bose particle. In this limit we describe the correlation functions by the vacuum expectation value of a boson-valued solution for Maxwell–Bloch differential equation. We evaluate long-distance asymptotics of the correlation functions in the small-mass limit.

## 1. Introduction

In this paper we consider correlation functions of exactly solvable models. Our approach is based on the determinant representation of quantum correlation functions [1]. We consider the thermodynamics of Bose gas with delta-interaction at finite temperature $T > 0$. The one-dimensional Bose gas with delta-function interaction is described by the canonical Bose fields $\psi(x)$ and $\psi^+(x)$ with the commutation relations:

$$[\psi(x), \psi^+(y)] = \delta(x-y) \qquad [\psi(x), \psi(y)] = [\psi^+(x), \psi^+(y)] = 0. \tag{1.1}$$

The Hamiltonian of the model is

$$H = \int \mathrm{d}x \, \left( \frac{1}{2m} \frac{\partial}{\partial x} \psi^+(x) \frac{\partial}{\partial x} \psi(x) + g\psi^+(x)\psi^+(x)\psi(x)\psi(x) - h\psi^+(x)\psi(x) \right) \tag{1.2}$$

where $m > 0$ is the mass, $g > 0$ is the coupling constant and $h > 0$ is the chemical potential. The Hamiltonian $H$ acts on the Fock space with the vacuum vector $|\mathrm{vac}\rangle$. The vacuum vector $|\mathrm{vac}\rangle$ is characterized by the relation:

$$\psi(x)|\mathrm{vac}\rangle = 0. \tag{1.3}$$

The dual vacuum vector $\langle\mathrm{vac}|$ is characterized by the relations:

$$\langle\mathrm{vac}|\psi^+(x) = 0 \qquad \langle\mathrm{vac} \mid \mathrm{vac}\rangle = 1. \tag{1.4}$$

The corresponding equation of motion

$$\mathrm{i}\frac{\partial}{\partial t}\psi = [\psi, H] = -\frac{1}{2m}\frac{\partial^2}{\partial x^2}\psi + 2g\psi^+\psi\psi - h\psi \tag{1.5}$$

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is called the quantum nonlinear Schrödinger equation in one space and one time dimension. The quantum field theory problem is reduced to a quantum mechanics problem. It is well known that in the $N$-particle sector the eigenvalue problem $H|\psi_N\rangle = E_N|\psi_N\rangle$, is equivalent to that described by the quantum mechanics $N$-body Hamiltonian

$$H_N = -\frac{1}{2m}\sum_{j=1}^{N}\frac{\partial^2}{\partial z_j^2} + 2g\sum_{1\leqslant j<k\leqslant N}\delta(z_k - z_j) - Nh. \tag{1.6}$$

Lieb and Linger [2] solved the eigenvalue problem $H_N\psi_N = E_N\psi_N$. They constructed the eigenfunctions $\psi_N = \psi_N(z_1,\ldots,z_N|\lambda_1,\ldots,\lambda_N)$ by means of the Bethe ansatz. The eigenfunction $\psi_N = \psi_N(z_1,\ldots,z_N|\lambda_1,\ldots,\lambda_N)$ depends on the spectral parameter $\lambda_1 < \cdots < \lambda_N$. The spectral parameters $\lambda_1 < \cdots < \lambda_N$ are determined by the periodic boundary conditions:

$$\psi_N(z_1,\ldots,z_j+L,\ldots,z_N|\lambda_1,\ldots,\lambda_N) = \psi_N(z_1,\ldots,z_j,\ldots,z_N|\lambda_1,\ldots,\lambda_N) \tag{1.7}$$

which amounts to the Bethe ansatz equations:

$$\mathrm{e}^{\mathrm{i}\lambda_j L} = -\prod_{k=1}^{N}\frac{\lambda_j - \lambda_k + 2\mathrm{i}mg}{\lambda_j - \lambda_k - 2\mathrm{i}mg} \qquad j = 1,\ldots,N. \tag{1.8}$$

Here $L > 0$ is the size of the box. The eigenvalue of the Hamiltonian $H_N$ is given by

$$E_N = \sum_{j=1}^{N}\left(\frac{1}{2m}\lambda_j^2 - h\right). \tag{1.9}$$

Lieb and Linger [2, 3] discussed the zero temperature thermodynamic limit. The ground state and its excitations are described by linear integral equations. Yang and Yang [4] discussed the finite-temperature thermodynamic limit. The state of thermodynamic equilibrium is described by nonlinear integral equations. The density of particles $\rho_\mathrm{p}(\lambda)$ and the density of holes $\rho_\mathrm{h}(\lambda)$ are described by the following nonlinear integral equations:

$$2\pi\rho_\mathrm{t}(\lambda) = 1 + \int_{-\infty}^{\infty} K(\lambda,\mu)\rho_\mathrm{p}(\mu)\,\mathrm{d}\mu \tag{1.10}$$

$$D = \frac{N}{L} = \int_{-\infty}^{\infty}\rho_\mathrm{p}(\mu)\,\mathrm{d}\mu \tag{1.11}$$

$$\varepsilon(\lambda) = \frac{\lambda^2}{2m} - h - \frac{T}{2\pi}\int_{-\infty}^{\infty} K(\lambda,\mu)\ln\left(1+\mathrm{e}^{-\varepsilon(\mu)/T}\right)\,\mathrm{d}\mu \tag{1.12}$$

where $T > 0$ is temperature and $D = N/L$ is the density of particles. Here the functions $\varepsilon(\lambda)$ and $\rho_\mathrm{t}(\lambda)$ are defined by

$$\frac{\rho_\mathrm{h}(\lambda)}{\rho_\mathrm{p}(\lambda)} = \mathrm{e}^{\varepsilon(\lambda)/T} \qquad \rho_\mathrm{t}(\lambda) = \rho_\mathrm{p}(\lambda) + \rho_\mathrm{h}(\lambda). \tag{1.13}$$

The integral kernel $K(\lambda,\mu)$ is defined by

$$K(\lambda,\mu) = \frac{4mg}{(\lambda-\mu)^2 + (2mg)^2}. \tag{1.14}$$

Consider the local density operator $j(x) = \psi^+(x)\psi(x)$. In this paper we consider the mean value of the operator

$$\exp\left(\alpha Q(x)\right). \tag{1.15}$$

Here $\alpha$ is an arbitrary complex parameter and $Q(x)$ is the operator of the number of particles on the interval $[0, x]$:

$$Q(x) = \int_0^x \psi^+(y)\psi(y)\,\mathrm{d}y. \tag{1.16}$$

We are interested in the generating function of the temperature-dependent correlation function defined by

$$\langle \exp(\alpha Q(x))\rangle_T = \frac{\mathrm{tr}\,(\exp(-H/T)\exp(\alpha Q(x)))}{\mathrm{tr}\,(\exp(-H/T))}. \tag{1.17}$$

The expectation value $\langle \exp(\alpha Q(x))\rangle_T$ is a remarkable quantity, because many interesting correlation functions can be extracted from $\langle \exp(\alpha Q(x))\rangle_T$. For example, the density correlation function

$$\langle j(x)j(0)\rangle_T = \frac{\mathrm{tr}\,(\exp(-H/T)\,j(x)j(0))}{\mathrm{tr}\,(\exp(-H/T))} \tag{1.18}$$

can be derived by

$$\langle j(x)j(0)\rangle_T = \frac{1}{2}\frac{\partial^2}{\partial x^2}\langle Q(x)^2\rangle_T = \frac{1}{2}\frac{\partial^2}{\partial x^2}\,\frac{\partial^2}{\partial \alpha^2}\langle \exp(\alpha Q(x))\rangle_T\bigg|_{\alpha=0}. \tag{1.19}$$

In this paper we are interested in the small-mass limit of the Bose particle:

$$m \to 0,\quad g \to \infty \qquad \text{such that the product} \quad c = 2mg \quad \text{is fixed.} \tag{1.20}$$

We want to emphasize that the small-mass limit is not a free-fermionic limit. The scattering matrix of the particles $\lambda_{\mathrm{p}}$ and $\lambda_{\mathrm{h}}$ is equal to

$$S(\lambda_{\mathrm{p}}, \lambda_{\mathrm{h}}) = \exp\left(-\mathrm{i}\delta(\lambda_{\mathrm{p}}, \lambda_{\mathrm{h}})\right) \qquad \lambda_{\mathrm{p}} > \lambda_{\mathrm{h}} \tag{1.21}$$

where the scattering phase $\delta$ satisfies the following integral equation:

$$\delta(\lambda_{\mathrm{p}}, \lambda_{\mathrm{h}}) - \frac{1}{2\pi}\int_{-\infty}^{\infty} K(\lambda_{\mathrm{p}}, \mu)\vartheta(\mu)\delta(\mu, \lambda_{\mathrm{h}}) = \mathrm{i}\ln\left(\frac{\mathrm{i}c + \lambda_{\mathrm{p}} - \lambda_{\mathrm{h}}}{\mathrm{i}c - \lambda_{\mathrm{p}} + \lambda_{\mathrm{h}}}\right). \tag{1.22}$$

Here we used

$$\vartheta(\lambda) = \frac{1}{1 + \mathrm{e}^{\varepsilon(\lambda)/T}} = \frac{\rho_{\mathrm{p}}(\lambda)}{\rho_{\mathrm{t}}(\lambda)}. \tag{1.23}$$

Therefore the small-mass limit is not a free-fermionic limit. In the small-mass limit we will show that the expectation value $\langle \exp(\alpha Q(x))\rangle_T$ is described by the vacuum expectation value of a boson-valued solution of the Maxwell–Bloch equation [5]. The plan of this paper is as follows. In section 2 we summarize known results of determinant representations for correlation functions. In section 3 we consider the small-mass limit of temperature correlation functions. The determinant representation for correlation functions simplifies in the small-mass limit. In section 4 we show that correlation functions can be described by the vacuum expectation value of a boson-valued solution of Maxwell–Bloch equation, in the small-mass limit. In section 5 we evaluate asymptotics of the correlation functions in the small-mass limit.

## 2. Determinant representation with dual fields

The purpose of this section is to summarize the known results of the determinant representation for temperature correlation functions [1]. First, we introduce the dual fields $\phi_j(\lambda)$, $(j = 1, \ldots, 4)$ defined by

$$\phi_j(\lambda) = p_j(\lambda) + q_j(\lambda) \qquad j = 1, \ldots, 4). \tag{2.1}$$

Here the fields $p_j(\lambda)$ and $q_j(\lambda)$ are defined by the commutation relations

$$\left.\begin{array}{l} [p_j(\lambda), p_k(\mu)] = [q_j(\lambda), q_k(\mu)] = 0 \\ [p_j(\lambda), q_k(\mu)] = H_{j,k}(\lambda, \mu) \end{array}\right\} \qquad (j, k = 1, \ldots, 4). \tag{2.2}$$

Here we used

$$H_{j,k}(\lambda, \mu) = \begin{pmatrix} -1 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & -1 & 1 \\ 0 & -1 & 1 & -1 \end{pmatrix}_{j,k} \ln(h(\lambda, \mu))$$

$$+ \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 \\ 0 & 1 & -1 & 1 \\ -1 & 0 & 1 & -1 \end{pmatrix}_{j,k} \ln(h(\mu, \lambda)) \tag{2.3}$$

where

$$h(\lambda, \mu) = \frac{1}{\mathrm{i}c}(\lambda - \mu + \mathrm{i}c). \tag{2.4}$$

The dual fields $\phi_j(\lambda)$ commute:

$$[\phi_j(\lambda), \phi_k(\mu)] = 0 \qquad (j, k = 1, \ldots, 4). \tag{2.5}$$

We introduce the auxiliary Fock space with the auxiliary vacuum vector $|0\rangle$. The auxiliary vacuum vector $|0\rangle$ is characterized by

$$p_j(\lambda)|0\rangle = 0 \qquad (j = 1, \ldots, 4). \tag{2.6}$$

The auxiliary dual vacuum $\langle 0|$ is characterized by

$$\langle 0|q_j(\lambda) = 0 \qquad (j = 1, \ldots, 4) \qquad \langle 0 \mid 0\rangle = 1. \tag{2.7}$$

We want to emphasize that the dual fields $\phi_j(\lambda)$ $(j = 1, \ldots, 4)$ and the auxiliary Fock space can be written in terms of the four standard Bose fields $\psi_j(\lambda)$, $\psi_j^+(\mu)$, $(j = 1, \ldots, 4)$ and the standard Fock vacuum $|0\rangle$ and the dual Fock vacuum $\langle 0|$:

$$\left.\begin{array}{l} [\psi_j(\lambda), \psi_k^+(\mu)] = \delta_{j,k}\delta(\lambda - \mu) \\ [\psi_j(\lambda), \psi_k(\mu)] = [\psi_j^+(\lambda), \psi_k^+(\mu)] = 0 \end{array}\right\} \qquad (j, k = 1, \ldots, 4) \tag{2.8}$$

$$\psi_j(\lambda)|0\rangle = 0 \qquad \langle 0|\psi_j^+(\lambda) = 0. \tag{2.9}$$

Actually, the dual fields can be realized by

$$p_j(\lambda) = \psi_j(\lambda) \qquad q_k(\mu) = \sum_{l=1}^{4} \int_{-\infty}^{\infty} H_{l,k}(\nu, \mu)\psi_l^+(\nu)\, \mathrm{d}\nu \qquad (j, k = 1, \ldots, 4). \tag{2.10}$$

Next we prepare two integral operators $\hat{V}_T$ and $\hat{K}_T$. The integral operator $\hat{V}_T$ is defined by

$$\left(\hat{V}_T f\right)(\lambda) = \int_{-\infty}^{\infty} V_T(\lambda, \mu) f(\mu)\, \mathrm{d}\mu. \tag{2.11}$$

The integral kernel $V_T(\lambda, \mu)$ is defined by product $V_T(\lambda, \mu) = V(\lambda, \mu)\vartheta(\mu)$. The first factor $V(\lambda, \mu)$ is defined by

$$\begin{aligned} V(\lambda, \mu) = \frac{1}{c}\Big\{ & t(\lambda, \mu) + t(\mu, \lambda) \exp\left(-\mathrm{i}x(\lambda - \mu) + \phi_1(\mu) - \phi_1(\lambda)\right) \\ & + \exp\left(\alpha + \phi_3(\lambda) + \phi_4(\mu)\right) \\ & \times \left(t(\mu, \lambda) + t(\lambda, \mu) \exp\left(-\mathrm{i}x(\lambda - \mu) + \phi_2(\lambda) - \phi_2(\mu)\right)\right)\Big\} \end{aligned} \tag{2.12}$$

where

$$t(\lambda, \mu) = \frac{(\mathrm{i}c)^2}{(\lambda - \mu)(\lambda - \mu + \mathrm{i}c)}. \tag{2.13}$$

We call the second factor $\vartheta(\lambda)$ the Fermi weight:

$$\vartheta(\lambda) = \frac{1}{1 + \mathrm{e}^{\varepsilon(\lambda)/T}} = \frac{\rho_{\mathrm{p}}(\lambda)}{\rho_{\mathrm{t}}(\lambda)}. \tag{2.14}$$

Because the dual fields $\phi_j(\lambda)$ commute with each other, we can define the quantity $\det\left(1 + (1/2\pi)\hat{V}_T\right)$. The integral operator $\hat{K}_T$ is defined by

$$\left(\hat{K}_T f\right)(\lambda) = \int_{-\infty}^{\infty} K_T(\lambda, \mu) f(\mu)\, \mathrm{d}\mu. \tag{2.15}$$

The integral kernel $K_T(\lambda, \mu)$ is defined by $K_T(\lambda, \mu) = K(\lambda, \mu)\vartheta(\mu)$. $K(\lambda, \mu)$ is defined in (1.14). Now we state the results which we will use in the following sections.

*Theorem 2.1 (Korepin [6]).* In terms of the dual fields $\phi_j(\lambda)$ $(j = 1, \ldots, 4)$, we can express the expectation value $\langle \exp(\alpha Q(x)) \rangle_T$ by the Fredholm determinant:

$$\langle \exp(\alpha Q(x)) \rangle_T = \frac{\langle 0 | \det\left(1 + (1/2\pi)\hat{V}_T\right) | 0 \rangle}{\det\left(1 - (1/2\pi)\hat{K}_T\right)}. \tag{2.16}$$

Here the symbol $\det\left(1 + (1/2\pi)\hat{V}_T\right)$ represents the Fredholm determinant corresponding to the following Fredholm integral equation of the second kind:

$$\left(\left(1 + \frac{1}{2\pi}\hat{V}_T\right) f\right)(\lambda) = g(\lambda) \qquad \text{for } \lambda \in (-\infty, \infty). \tag{2.17}$$

The denominator $\det\left(1 - (1/2\pi)\hat{K}_T\right)$ represents the Fredholm determinant corresponding to the following Fredholm integral equation of the second kind:

$$\left(\left(1 - \frac{1}{2\pi}\hat{K}_T\right) f\right)(\lambda) = g(\lambda) \qquad \text{for } \lambda \in (-\infty, \infty). \tag{2.18}$$

## 3. The small-mass limit of the Bose particle

In this section we will show that in the small-mass limit: $m \to 0$, $g \to \infty$, such that $c = 2mg$ is fixed, a simplification occurs. As explained in the introduction, the scattering matrix depends on the product $c = 2mg$, and not just on $g$. Therefore the limit of small mass is not a free-fermion limit. We want to emphasize this point. In the sequel we consider the limit of small mass. First we evaluate the solution of the Yang–Yang equation.

$$\varepsilon(\lambda) = \frac{\lambda^2}{2m} - h - \frac{T}{2\pi}\int_{-\infty}^{\infty} K(\lambda,\mu)\ln\left(1+\mathrm{e}^{-\varepsilon(\mu)/T}\right)\,\mathrm{d}\mu. \tag{3.1}$$

This is done following [4].

*Lemma 3.1.* In the small-mass limit of the Bose particle, a solution of the Yang–Yang equation (3.1) is evaluated as

$$\varepsilon(\lambda) = \frac{\lambda^2}{2m} - h + \mathrm{O}(\sqrt{m}). \tag{3.2}$$

*Proof.* In [4] Yang and Yang derived the following inequalities:

$$\frac{\lambda^2}{2m} - h \geqslant \varepsilon(\lambda) \geqslant \frac{\lambda^2}{2m} + x_0 \tag{3.3}$$

where $x_0$ is defined by the integral equation

$$x_0 = -h - \frac{T}{2\pi}\int_{-\infty}^{\infty} K(0,\mu)\ln\left(1+\exp\left(-\frac{1}{T}\left(\frac{\mu^2}{2m}+x_0\right)\right)\right)\,\mathrm{d}u. \tag{3.4}$$

The existence of $x_0$ is proved in [4]. Let us change the integration variable to $\nu = \mu/\sqrt{2m}$. In the limit of small mass, $\sqrt{cg}$ tends to $\infty$. Therefore we obtain

$$x_0 = -h - \frac{T}{\pi}\int_{-\infty}^{\infty}\frac{\sqrt{cg}}{(\sqrt{cg})^2+\nu^2}\ln\left(1+\mathrm{e}^{-(\nu^2+x_0)/T}\right)\,\mathrm{d}\nu \tag{3.5}$$

$$= -h + x_0 - \frac{T}{\pi}\int_{-\infty}^{\infty}\frac{\sqrt{cg}}{(\sqrt{cg})^2+\nu^2}\ln\left(\mathrm{e}^{x_0/T}+\mathrm{e}^{-\nu^2/T}\right)\,\mathrm{d}\nu \tag{3.6}$$

$$= -h - \frac{T}{\pi}\frac{1}{\sqrt{cg}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\ln\left(1+\mathrm{e}^{-(\nu^2+x_0)/T}\right)\,\mathrm{d}\nu + \mathrm{O}(m). \tag{3.7}$$

When we assume $|x_0| \to \infty$, this contradicts (3.6). Therefore we can assume that $|x_0|$ is bounded. Therefore, from equation (3.7), we can deduce $x_0 = -h + \mathrm{O}(\sqrt{m})$. □

From lemma 3.1, we can evaluate the Fermi weight $\vartheta(\lambda)$. The Fermi weight $\vartheta(\lambda)$ has a very sharp maximum at $\lambda = 0$, from which it decreases very rapidly to 0. Therefore a simplification occurs. First we consider the dual fields. In what follows we consider the case where the spectral parameters are restricted to $\lambda, \mu \approx \mathrm{O}(\sqrt{m})$. We observe the simplification of the commutation relations:

$$[p_j(\lambda), q_k(\mu)] = \begin{pmatrix} 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix}_{j,k} \frac{\mathrm{i}}{c}(\mu-\lambda) + \mathrm{O}(m). \tag{3.8}$$

Therefore we can identify pairs of fields:

$$p_1(\lambda) = -p_2(\lambda) \qquad p_3(\lambda) = p_4(\lambda) \tag{3.9}$$

$$q_1(\lambda) = -q_2(\lambda) \qquad q_3(\lambda) = q_4(\lambda) \tag{3.10}$$

$$\phi_1(\lambda) = -\phi_2(\lambda) \qquad \phi_3(\lambda) = \phi_4(\lambda). \tag{3.11}$$

Furthermore, because the first term of the commutation relation (3.8) is a linear function of the spectral parameters, we can choose a representation of the fields such that $\phi_j(\lambda)$ are linear functions of the spectral parameter $\lambda$:

$$\left.\begin{aligned} \phi_j(\lambda) &= \phi_j(0) + \phi_j'(0)\lambda \\ \phi_j(0) &= p_j(0) + q_j(0) \\ \phi_j'(0) &= p_j'(0) + q_j'(0) \end{aligned}\right\} \qquad (j = 1, 3). \tag{3.12}$$

Here the commutation relations are

$$[p_j(0), q_k(0)] = 0 = [p_j'(0), q_k'(0)] \qquad (j, k = 1, 3) \tag{3.13}$$

$$[p_1'(0), q_3(0)] = \frac{\mathrm{i}}{c} = -[p_3'(0), q_1(0)] \qquad [p_3(0), q_1'(0)] = \frac{\mathrm{i}}{c} = -[p_1(0), q_3'(0)]. \tag{3.14}$$

The actions on the auxiliary vacuum are

$$p_1(0)|0\rangle = p_3(0)|0\rangle = p_1'(0)|0\rangle = p_3'(0)|0\rangle = 0 \tag{3.15}$$

$$\langle 0|q_1(0) = \langle 0|q_3(0) = \langle 0|q_1'(0) = \langle 0|q_3'(0) = 0. \tag{3.16}$$

Furthermore, we arrive at the following formula.

*Theorem 3.2.* In the small-mass limit of the Bose particle, the expectation value of the Fredholm determinant simplifies as follows:

$$\langle 0|\det\left(1 + \frac{1}{2\pi}\hat{V}_T\right)|0\rangle \longmapsto \langle 0|\det\left(1 + \hat{V}_{0,T}\right)|0\rangle. \tag{3.17}$$

Here the symbol $\hat{V}_{0,T}$ is the integral operator defined by

$$\left(\hat{V}_{0,T} f\right)(\lambda) = \int_{-\infty}^{\infty} V_{0,T}(\lambda, \mu) f(\mu)\,\mathrm{d}\mu \tag{3.18}$$

where the integral kernel is defined by product

$$V_{0,T}(\lambda, \mu) = \left(\frac{\mathrm{e}^{\hat{\alpha}} - 1}{\pi}\right) \frac{\sin\frac{1}{2}\hat{x}(\lambda - \mu)}{\lambda - \mu} \vartheta_0\left(\frac{\mu}{\sqrt{2mT}}, \frac{h}{T}\right) \tag{3.19}$$

where

$$\vartheta_0(\mu, \beta) = \frac{1}{1 + \mathrm{e}^{\mu^2 - \beta}}. \tag{3.20}$$

Here we used the abbreviations

$$\hat{\alpha} = \alpha + \hat{\alpha}_p + \hat{\alpha}_q \qquad \hat{x} = x + \hat{x}_p + \hat{x}_q \tag{3.21}$$

$$\hat{\alpha}_p = 2p_3(0) \qquad \hat{\alpha}_q = 2q_3(0) \qquad \hat{x}_p = -\mathrm{i}p_1'(0) \qquad \hat{x}_q = -\mathrm{i}q_1'(0). \tag{3.22}$$

The commutation relations and the actions on the auxiliary vacuum become

$$[\hat{x}_p, \hat{\alpha}_q] = \frac{2}{c} \qquad [\hat{\alpha}_p, \hat{x}_q] = \frac{2}{c} \tag{3.23}$$

$$\hat{x}_p|0\rangle = 0 = \hat{\alpha}_p|0\rangle \qquad \langle 0|\hat{x}_q = 0 = \langle 0|\hat{\alpha}_q. \tag{3.24}$$

The dual fields $\hat{\alpha}$ and $\hat{x}$ commute with each other: $[\hat{\alpha}, \hat{x}] = 0$.

*Proof.* From lemma 3.1, the Fermi weight $\vartheta(\lambda)$ has a very sharp maximum at $\lambda = 0$ and decreases very rapidly to 0. When we consider the integral operator $\hat{V}_T$, we can restrict our consideration to the case of the spectral parameters $\lambda, \mu \approx \mathrm{O}(\sqrt{m})$. Therefore we can use the above dual fields simplification. We can identify four dual fields to two dual fields, which are linear in the spectral parameters $\lambda, \mu$. Furthermore, since the relations $[p_3'(0), \phi_1(\lambda) - \phi_1(\mu)] = 0$, $[q_3'(0), \phi_1(\lambda) - \phi_1(\mu)] = 0$ and $\langle 0|q_3'(0) = 0$, $p_3'(0)|0\rangle = 0$ hold, we can drop $p_3'(0)$, $q_3'(0)$ in the expectation value $\langle 0|\det\big(1 + (1/2\pi)\hat{V}_T\big)|0\rangle$. Next we perform a similarity transformation $\exp\big(\frac{1}{2}\mathrm{i}\lambda(x - \mathrm{i}\phi_1'(0))\big)$ which leaves the Fredholm determinant invariant. Finally we substitute the Fermi weight $\vartheta(\mu)$ by the modified Fermi weight $\vartheta_0\big(\mu/\sqrt{2mT}, h/T\big)$. We get the desired formula. □

The denominator of the expectation value (2.16) becomes the following one:

$$\det\left(1 - \frac{1}{2\pi}\hat{K}_T\right) = 1 - \frac{\sqrt{2T}}{\pi c}\, d\left(\frac{h}{T}\right)\sqrt{m} + \mathrm{O}(m) \tag{3.25}$$

where we used

$$d(\beta) = \int_{-\infty}^{\infty} \vartheta_0(\mu, \beta)\, \mathrm{d}\mu. \tag{3.26}$$

The density $D$ can be written as

$$D = \frac{N}{L} = \int_{-\infty}^{\infty} \rho_{\mathrm{p}}(\mu)\, \mathrm{d}\mu = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{1 + \exp\left(\frac{1}{T}\left(\frac{\mu^2}{2m} - h\right)\right)}\, \mathrm{d}\mu + \mathrm{O}(m). \tag{3.27}$$

Therefore we can write

$$\det\left(1 - \frac{1}{2\pi}\hat{K}_T\right) = 1 - \frac{2}{c}D + \mathrm{O}(m). \tag{3.28}$$

Therefore we arrive at the simplified formula for correlation functions.

*Corollary 3.3.* In the small-mass limit of the Bose particle, the temperature correlation function simplifies as follows:

$$\langle \exp(\alpha Q(x))\rangle_T \longmapsto \langle 0|\det\left(1 + \hat{V}_{0,T}\right)|0\rangle\left(1 + \frac{2}{c}D\right). \tag{3.29}$$

Here $D = N/L$ is the density of the thermodynamic limit.

## 4. Maxwell–Bloch differential equation

In this section we consider the differential equation for the temperature correlation function in the small-mass limit of the Bose particle. In the small-mass limit, the Fredholm determinant $\det\big(1 + \hat{V}_{0,T}\big)$ is a $\tau$-function of the Maxwell–Bloch equation, taking values in a commutative subalgebra of the quantum operator algebra. It can easily be seen that after introducing new variables, the auxiliary field $\hat{y}$ and the scaled chemical potential $\beta$,

$$\hat{y} = y + \hat{y}_p + \hat{y}_q \qquad y = \sqrt{\frac{mT}{2}}x \qquad \hat{y}_p = \sqrt{\frac{mT}{2}}\hat{x}_p \qquad \hat{y}_q = \sqrt{\frac{mT}{2}}\hat{x}_q \qquad \beta = \frac{h}{T} \tag{4.1}$$

the Fredholm determinant $\det\big(1 + \hat{V}_{0,T}\big)$ can be rewritten, after the corresponding change $\lambda \to \lambda/\sqrt{2mT}$ of the spectral parameter, as

$$\det\left(1 + \hat{V}_{0,T}\right) = \det\left(1 - \hat{\gamma}\hat{W}\right)\Big|_{\hat{\gamma} = (1 - \exp(\hat{\alpha}))/\pi}. \tag{4.2}$$

We want to emphasize that $\hat{y}$ is an operator in the auxiliary space. The integral operator $\hat{W}$ is defined by

$$\left(\hat{W} f\right)(\lambda) = \int_{-\infty}^{\infty} W(\lambda, \mu) f(\mu)\, \mathrm{d}\mu \tag{4.3}$$

where the integral kernel $W(\lambda, \mu)$ is given by

$$W(\lambda, \mu) = \frac{\sin \hat{y}(\lambda - \mu)}{\lambda - \mu} \vartheta_0(\mu, \beta). \tag{4.4}$$

The algebraic structure of the Fredholm determinant $\det(1 - \hat{\gamma}\hat{W})|_{\hat{\gamma}=(1-\exp(\hat{\alpha}))/\pi}$ has been investigated in the context of correlation functions for the impenetrable Bose gas [1]. It is convenient to introduce the function $\sigma$ defined by

$$\sigma\left(\hat{y}, \beta, \hat{\alpha}\right) = \ln\det\left(1 - \hat{\gamma}\hat{W}\right)\Big|_{\hat{\gamma}=(1-\exp(\hat{\alpha}))/\pi}. \tag{4.5}$$

The operator $\sigma$ satisfies the Maxwell–Bloch equation in the case that $\hat{y}$ and $\hat{\alpha}$ are real numbers [1]. In our case, $\hat{y}$ and $\hat{\alpha}$ are quantum operator, but due to the fact that they commute with each other, we can follow the derivation in [1]. Therefore we arrive at the following results. In what follows we use the following operator-derivation notation:

$$\frac{\partial}{\partial \hat{y}} F(\hat{y}) := \frac{\partial}{\partial z} F(z)\Big|_{z=\hat{y}} \tag{4.6}$$

where $F = F(z)$ is a function of $z$.

*Proposition 4.1.* The operator $\sigma(\hat{y}, \beta, \hat{\alpha}) = \ln\det(1 - \hat{\gamma}\hat{W})|_{\hat{\gamma}=(1-\exp(\hat{\alpha}))/\pi}$ obeys the following nonlinear partial differential equation:

$$\left(\frac{\partial}{\partial\beta}\frac{\partial^2}{\partial\hat{y}^2}\sigma\right)^2 = -4\left(\frac{\partial^2}{\partial\hat{y}^2}\sigma\right)\left(2\hat{y}\frac{\partial}{\partial\beta}\frac{\partial}{\partial\hat{y}}\sigma + \left(\frac{\partial}{\partial\beta}\frac{\partial}{\partial\hat{y}}\sigma\right)^2 - 2\frac{\partial}{\partial\beta}\sigma\right) \tag{4.7}$$

with the initial conditions

$$\sigma = -\left(\frac{1-\mathrm{e}^{\hat{\alpha}}}{\pi} d(\beta)\right)\hat{y} - \left(\frac{1-\mathrm{e}^{\hat{\alpha}}}{\pi} d(\beta)\right)^2 \frac{\hat{y}^2}{2} + \mathrm{O}(\hat{y}^3) \tag{4.8}$$

$$\lim_{\beta\to-\infty} \sigma(\hat{y}, \beta, \hat{\alpha}) = 0 \tag{4.9}$$

where the scalar function $d(\beta)$ is defined in (3.26).

These initial data fix the solution uniquely. The nonlinear differential equation (4.7) is called the Maxwell–Bloch equation [5]. Algebraically, it is known that at $T = 0$ the operator $\sigma$ depends only on product of variables $\hat{y}\sqrt{\beta}$ [6]. We set $\tau = \hat{y}\sqrt{\beta} = \sqrt{mh/2}\,\hat{x}$. Equation (4.7) is rewritten at $T = 0$ for the operator

$$\sigma_0(\tau) = \tau\frac{\mathrm{d}}{\mathrm{d}\tau}\ln\det\left(1 - \hat{\gamma}\hat{W}\right)\Big|_{\hat{\gamma}=(1-\exp(\hat{\alpha}))/\pi} \tag{4.10}$$

as

$$\left(\tau\frac{\mathrm{d}^2\sigma_0}{\mathrm{d}\tau^2}\right)^2 = -4\left(\tau\frac{\mathrm{d}\sigma_0}{\mathrm{d}\tau} - \sigma_0\right)\left(4\tau\frac{\mathrm{d}\sigma_0}{\mathrm{d}\tau} + \left(\frac{\mathrm{d}\sigma_0}{\mathrm{d}\tau}\right)^2 - 4\sigma_0\right). \tag{4.11}$$

This ordinary differential equation is the fifth Painlevé equation in [7]. Actually, rewriting (4.11) in terms of the function $y_0(\tau)$ defined by

$$\sigma_0(\tau) = -4\mathrm{i}\tau u_0(\tau) + \frac{u_0(\tau)^2}{y_0(\tau)}(y_0(\tau) - 1)^2 \qquad u_0(\tau) = \frac{4\mathrm{i}\tau y_0(\tau) - \tau\,\mathrm{d}y_0(\tau)/\mathrm{d}\tau}{2(y_0(\tau) - 1)^2} \tag{4.12}$$

we can get the familiar formula of the fifth Painlevé differential equation for the function $w(\tau) = y_0(\frac{1}{2}\tau)$:

$$\frac{\mathrm{d}^2 w}{\mathrm{d}\tau^2} = \left(\frac{\mathrm{d}w}{\mathrm{d}\tau}\right)^2 \frac{3w-1}{2w(w-1)} + \frac{2w(w+1)}{w-1} + \frac{2\mathrm{i}w}{\tau} - \frac{1}{\tau}\frac{\mathrm{d}w}{\mathrm{d}\tau}. \tag{4.13}$$

Next we derive the asymptotics of $\sigma(\hat{y}, \beta, \hat{\alpha}) = \ln\det(1 - \hat{\gamma}\hat{W})|_{\hat{\gamma}=(1-\exp(\hat{\alpha}))/\pi}$. By means of the Riemann–Hilbert method, the asymptotics of $\sigma$ are derived for the case where $\hat{y}$ and $\hat{\alpha}$ are real numbers [1]. The idea of the Riemann–Hilbert method is due to Professor A R Its. In our case, $\hat{y}$ and $\hat{\alpha}$ are quantum operators, but due to the fact that they commute, we can follow the derivation in [1]. We arrive at the following asymptotics.

*Proposition 4.2.* The asymptotics of the operator $\sigma(\hat{y}, \beta, \hat{\alpha})$ for large $\hat{y}$ become the following:

$$\begin{aligned}\sigma(\hat{y}, \beta, \hat{\alpha}) = {} & -\hat{y}\; C(\beta, \hat{\alpha}) + \frac{1}{2}\int_{-\infty}^{\beta} \left(\frac{\partial C(b, \hat{\alpha})}{\partial b}\right)^2 \mathrm{d}b \\ & - \frac{1}{8}\frac{(\mathrm{e}^{-\hat{\alpha}} - 1)^2}{r_1(\hat{\alpha})^4\; |a(\lambda_1(\hat{\alpha}), \hat{\alpha})|^4} \exp\left(-4r_1(\hat{\alpha})\sin\varphi_1(\hat{\alpha})\;\hat{y}\right) \\ & \times \left(\frac{1}{\sin^2\varphi_1(\hat{\alpha})} + \cos\left\{4\hat{y}r_1(\hat{\alpha})\cos\varphi_1(\hat{\alpha}) - 4\arg a(\lambda_1(\hat{\alpha}), \hat{\alpha}) - 4\varphi_1(\hat{\alpha})\right\}\right) \\ & + \mathrm{o}\left(\exp\left(-4r_1(\hat{\alpha})\sin\varphi_1(\hat{\alpha})\;\hat{y}\right)\right).\end{aligned} \tag{4.14}$$

Here we set

$$C(\beta, \alpha) = \frac{1}{\pi}\int_{-\infty}^{\infty} \ln\left(\frac{1 + \mathrm{e}^{\mu^2-\beta}}{\mathrm{e}^{\alpha} + \mathrm{e}^{\mu^2-\beta}}\right) \mathrm{d}\mu \tag{4.15}$$

$$\lambda_1(\alpha) = \sqrt{\alpha + \beta + \pi\mathrm{i}} \qquad r_1(\alpha) = |\lambda_1(\alpha)| \qquad \varphi_1(\alpha) = \arg\lambda_1(\alpha) \tag{4.16}$$

$$a(\lambda, \alpha) = \exp\left\{\frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty}\frac{\mathrm{d}\mu}{\mu - \lambda}\ln\left(\frac{1 + \mathrm{e}^{\mu^2-\beta}}{\mathrm{e}^{\alpha} + \mathrm{e}^{\mu^2-\beta}}\right)\right\}. \tag{4.17}$$

## 5. Evaluation of the mean value

In this section we evaluate the vacuum expectation value of the operator $\det(1 - \hat{\gamma}\hat{W})|_{\hat{\gamma}=(1-\exp(\hat{\alpha}))/\pi}$ for $y = \sqrt{mT/2}\,x \to +\infty$. From corollary 4.2, we deduce

$$\begin{aligned}\langle 0|\; \det\left(1 - \hat{\gamma}\hat{W}\right)\Big|_{\hat{\gamma}=(1-\exp(\hat{\alpha}))/\pi} |0\rangle \\ = \langle 0|A(\beta, \hat{\alpha})\mathrm{e}^{-C(\beta,\hat{\alpha})\hat{y}} + B(\beta, \hat{\alpha})\exp\left(-\{C(\beta, \hat{\alpha}) + 4r_1(\hat{\alpha})\sin\varphi_1(\hat{\alpha})\}\hat{y}\right) \\ + G\left(\beta, \hat{\alpha}\right)\exp\left(\{-C(\beta, \hat{\alpha}) + 4\mathrm{i}\lambda_1(\hat{\alpha})\}\hat{y}\right) \\ + H\left(\beta, \hat{\alpha}\right)\exp\left(\{-C(\beta, \hat{\alpha}) - 4\mathrm{i}\lambda_1^*(\hat{\alpha})\}\hat{y}\right)|0\rangle + \cdots.\end{aligned} \tag{5.1}$$

Here we set

$$A(\beta,\alpha) = \exp\left\{\frac{1}{2}\int_{-\infty}^{\beta}\left(\frac{\partial C(b,\alpha)}{\partial b}\right)^2 \mathrm{d}b\right\} \tag{5.2}$$

$$B(\beta,\alpha) = -\frac{(\mathrm{e}^{-\alpha}-1)^2}{8r_1(\alpha)^4\sin^2\varphi_1(\alpha)}\left(\frac{a(\lambda_1^*(\alpha),\alpha)}{a(\lambda_1(\alpha),\alpha)}\right)^2 A(\beta,\alpha) \tag{5.3}$$

$$G(\beta,\alpha) = -\frac{(\mathrm{e}^{-\alpha}-1)^2}{16}\frac{1}{\lambda_1(\alpha)^4 a(\lambda_1(\alpha),\alpha)^4}A(\beta,\alpha) \tag{5.4}$$

$$H(\beta,\alpha) = -\frac{(\mathrm{e}^{-\alpha}-1)^2}{16}\frac{a(\lambda_1^*(\alpha),\alpha)^4}{\lambda_1^*(\alpha)^4}A(\beta,\alpha) \tag{5.5}$$

where $C(\beta,\alpha), \lambda_1(\alpha), r_1(\alpha), \varphi_1(\alpha)$ and $a(\lambda,\alpha)$ are defined in (4.15), (4.16) and (4.17). $\lambda_1^*(\alpha)$ is the complex conjugation of $\lambda_1(\alpha)$, i.e.

$$\lambda_1^*(\alpha) = \sqrt{\alpha+\beta-\pi\mathrm{i}}. \tag{5.6}$$

$H(\beta,\alpha)$ is the complex conjugation of $G(\beta,\alpha)$.

In this section we evaluate the right-hand of the above vacuum expectation value. For the convenience of the reader we summarize below the commutation relations of the quantum operators:

$$\hat{y} = y + \hat{y}_p + \hat{y}_q \qquad y = \sqrt{\frac{mT}{2}}x,\ \ \hat{y}_p = \sqrt{\frac{mT}{2}}\hat{x}_p \qquad \hat{y}_q = \sqrt{\frac{mT}{2}}\hat{x}_q \tag{5.7}$$

$$\hat{\alpha} = \alpha + \hat{\alpha}_p + \hat{\alpha}_q \qquad \beta = \frac{h}{T} \tag{5.8}$$

$$[\hat{y}_p, \hat{\alpha}_q] = \frac{\sqrt{2mT}}{c} = [\hat{\alpha}_p, \hat{y}_q] \qquad \hat{y}_p|0\rangle = 0 = \hat{\alpha}_p|0\rangle \qquad \langle 0|\hat{y}_q = 0 = \langle 0|\hat{\alpha}_q. \tag{5.9}$$

The following proposition is the key to calculating the vacuum expectation value.

*Proposition 5.1.* The following asymptotic formula holds at large $y \to +\infty$:

$$\langle 0|\mathrm{e}^{\hat{y}E(\hat{\alpha})}F(\hat{\alpha})|0\rangle = F\left(\alpha + \frac{\sqrt{2mT}}{c}E(\alpha)\right)\mathrm{e}^{yE(\alpha)} + \cdots . \tag{5.10}$$

Here $E(\alpha)$ and $F(\alpha)$ are meromorphic functions of $\alpha$.

*Proof.* In this proof we use the following abbreviations:

$$\delta = \frac{\sqrt{2mT}}{c} \qquad A_0 + A_1\hat{\alpha}_q + A_2\hat{\alpha_q}^2 + \cdots = E(\alpha+\hat{\alpha}_q). \tag{5.11}$$

First we expand the exponential function and use the relations $\langle 0|\hat{y}_q = 0$, $\hat{\alpha}_p|0\rangle = 0$ and $[\hat{\alpha}_p, \hat{\alpha}_q] = 0$. We obtain

$$\mathrm{EV} := \langle 0|F(\hat{\alpha})\exp\left\{\hat{y}E(\hat{\alpha})\right\}|0\rangle = \langle 0|\sum_{n=0}^{\infty}\frac{1}{n!}(y+\hat{y}_p)^n(E(\alpha+\hat{\alpha}_q))^n F(\hat{\alpha})|0\rangle. \tag{5.12}$$

We expand

$$(E(\alpha+\hat{\alpha}_q))^n = (A_0 + A_1\hat{\alpha}_q + A_2\hat{\alpha_q}^2 + \cdots)^n \tag{5.13}$$

and using the commutation relation

$$\langle 0|[f(\hat{y}_p), \hat{\alpha}_q^k] = \delta^k \langle 0| f^{(k)}(\hat{y}_p) \tag{5.14}$$

we obtain

$$\begin{aligned}\mathrm{EV} = \langle 0| \sum_{n=0}^{\infty} \frac{1}{n!}(y+\hat{y}_p)^n \sum_{\substack{m_0+m_1+m_2+\cdots=n \\ m_j \geqslant 0}} \frac{n!}{m_0! m_1! m_2! \cdots} A_0^{m_0} A_1^{m_1} A_2^{m_2} \cdots \\ \times \hat{\alpha}_q^{m_1+2m_2+3m_3+\cdots} F(\hat{\alpha})|0\rangle \end{aligned} \tag{5.15}$$

$$\begin{aligned} = \langle 0| \sum_{n=0}^{\infty} \sum_{\substack{m_0+m_1+m_2+\cdots=n \\ m_j \geqslant 0}} \frac{n!}{m_0! m_1! m_2! \cdots} \delta^{m_1+2m_2+3m_3+\cdots} \\ \times (y+\hat{y}_p)^{n-(m_1+2m_2+3m_3+\cdots)} \\ \times A_0^{m_0} A_1^{m_1} A_2^{m_2} \cdots \frac{n(n-1)\cdots(n+1-(m_1+2m_2+3m_3+\cdots))}{n!} F(\hat{\alpha})|0\rangle. \end{aligned} \tag{5.16}$$

Using the relation

$$\frac{1}{2\pi \mathrm{i}} \oint \frac{\mathrm{e}^t}{t^{n-k+1}}\, \mathrm{d}t = \frac{n(n-1)\cdots(n-k+1)}{n!} = \frac{1}{(n-k)!} \tag{5.17}$$

we can factor as follows:

$$\mathrm{EV} = \langle 0| \sum_{n=0}^{\infty} \frac{1}{2\pi \mathrm{i}} \oint \frac{\mathrm{e}^t}{t} \left( \frac{A_0(y+\hat{y}_p)}{t} + A_1\delta + \frac{A_2\delta^2 t}{y+\hat{y}_p} + \frac{A_3\delta^3 t^2}{(y+\hat{y}_p)^2} + \cdots \right)^n F(\hat{\alpha})|0\rangle \tag{5.18}$$

$$= \langle 0| \sum_{n=0}^{\infty} \frac{1}{2\pi \mathrm{i}} \oint \frac{\mathrm{e}^t}{t} \left( \frac{A_0(y+\hat{y}_p)}{t} \right)^n F(\hat{\alpha})|0\rangle + \cdots \qquad \text{for} \quad y \to +\infty. \tag{5.19}$$

Using the relations

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots \qquad f(z) = \frac{1}{2\pi \mathrm{i}} \oint \frac{f(t)}{t-z}\, \mathrm{d}t \tag{5.20}$$

we obtain the following:

$$\mathrm{EV} = \langle 0| \frac{1}{2\pi \mathrm{i}} \oint \frac{\mathrm{e}^t}{t - A_0(y+\hat{y}_p)} F(\hat{\alpha})|0\rangle + \cdots = \langle 0| \mathrm{e}^{A_0(y+\hat{y}_p)} F(\hat{\alpha})|0\rangle + \cdots. \tag{5.21}$$

Using the relation $\mathrm{e}^A B \mathrm{e}^{-A} = \mathrm{e}^{ad(A)}(B)$, we obtain

$$\mathrm{e}^{E(\alpha)\hat{y}_p} F(\hat{\alpha}) \mathrm{e}^{-E(\alpha)\hat{y}_p} = \mathrm{e}^{E(\alpha) ad(\hat{y}_p)} F(\hat{\alpha}) = \exp\left( E(\alpha) \frac{\sqrt{2mT}}{c} \frac{\partial}{\partial a} \right) F(a) \bigg|_{a=\hat{\alpha}}. \tag{5.22}$$

Therefore we can drop the quantum operators in the expectation value:

$$\begin{aligned} \mathrm{EV} = \mathrm{e}^{yE(\alpha)} \langle 0| \exp\left( E(\alpha) \frac{\sqrt{2mT}}{c} \frac{\partial}{\partial a} \right) F(a) \bigg|_{a=\hat{\alpha}} \mathrm{e}^{E(\alpha)\hat{y}_p} |0\rangle + \cdots \\ = \exp\left( E(\alpha) \frac{\sqrt{2mT}}{c} \frac{\partial}{\partial a} \right) F(a) \bigg|_{a=\alpha} + \cdots . \end{aligned} \tag{5.23}$$

Here we have used the relations $\langle 0|\hat{\alpha}_q = 0 = \hat{\alpha}_p|0\rangle$, $\hat{y}_p|0\rangle = 0$.

Because the exponential of derivation is a shift operator:

$$\exp\left(w\frac{\partial}{\partial z}\right) f(z) = f(z+w) \tag{5.24}$$

we arrive at (5.10). □

Now, we arrive at the following theorem.

*Theorem 5.2.* The leading terms of the asymptotics of the expectation value behave exponentially as follows.

$$\begin{aligned}\langle 0|\det\left(1-\hat{\gamma}\hat{W}\right)\Big|_{\hat{\gamma}=(1-\exp(\hat{\alpha}))/\pi}|0\rangle &= A\left(\beta, \alpha - \frac{\sqrt{2mT}}{c}C(\beta,\alpha)\right)\mathrm{e}^{-C(\beta,\alpha)y}\\ &\quad + B\left(\beta, \alpha - \frac{\sqrt{2mT}}{c}\{C(\beta,\alpha)+4r_1(\alpha)\sin\varphi_1(\alpha)\}\right)\\ &\quad \times \exp\left(-\{C(\beta,\alpha)+4r_1(\alpha)\sin\varphi_1(\alpha)\}y\right)\\ &\quad + G\left(\beta, \alpha + \frac{\sqrt{2mT}}{c}\{-C(\beta,\alpha)+4\mathrm{i}\lambda_1(\alpha)\}\right)\exp\left(\{-C(\beta,\alpha)+4\mathrm{i}\lambda_1(\alpha)\}y\right)\\ &\quad + H\left(\beta, \alpha + \frac{\sqrt{2mT}}{c}\{-C(\beta,\alpha)-4\mathrm{i}\lambda_1^*(\alpha)\}\right)\\ &\quad \times \exp\left(\{-C(\beta,\alpha)-4\mathrm{i}\lambda_1^*(\alpha)\}y\right) + \cdots .\end{aligned} \tag{5.25}$$

Here $A(\beta,\alpha)$, $B(\beta,\alpha)$, $G(\beta,\alpha)$ and $H(\beta,\alpha)$ are defined in (5.2), (5.3), (5.4) and (5.5).

*Proof.* Applying proposition 5.1 to (5.1), we arrive at the result. □

When we consider $c=\infty$, theorem 5.2 coincides with the asymptotics results for the impenetrable Bose gas case [1].

*Corollary 5.3.* In the limit $m\to 0$, $g\to\infty$, $x\to\infty$ such that $c=2mg$ fixed and $\sqrt{m}x\to\infty$, the leading terms of asymptotics of the expectation value become

$$\begin{aligned}\langle j(x)j(0)\rangle_T \to D^2 &+ \frac{mT}{2}\left(B_0(\beta)+B_1(\beta)\sqrt{\frac{mT}{2}}x + B_2(\beta)\left(\sqrt{\frac{mT}{2}}x\right)^2\right)\\ &\times \exp\left\{-4r_1(0)\sin\varphi_1(0)\sqrt{\frac{mT}{2}}x\right\}\\ &+ \frac{mT}{2}\left(G_0(\beta)+G_1(\beta)\sqrt{\frac{mT}{2}}x + G_2(\beta)\left(\sqrt{\frac{mT}{2}}x\right)^2\right)\\ &\times \exp\left\{4\mathrm{i}\lambda_1(0)\sqrt{\frac{mT}{2}}x\right\}\end{aligned}$$

$$+\frac{mT}{2}\left(H_0(\beta)+H_1(\beta)\sqrt{\frac{mT}{2}}x+H_2(\beta)\left(\sqrt{\frac{mT}{2}}x\right)^2\right)$$
$$\times\ \exp\left\{-4\mathrm{i}\lambda_1^*(0)\sqrt{\frac{mT}{2}}x\right\}+\cdots\ . \qquad (5.26)$$

Here $D=N/L$ is the density of the thermodynamics and $\beta=h/T$. Here $B_j(\beta),\ G_j(\beta)$ and $H_j(\beta),\ (j=0,1,2)$ are functions of $\beta$. $H_j(\beta)\ (j=0,1,2)$ is the complex conjugation of $G_j(\beta)\ (j=0,1,2)$, i.e. $H_j(\beta)=G_j^*(\beta)$. Explicit formulae for $B_j(\beta),\ G_j(\beta)$ and $H_j(\beta)\ (j=0,1,2)$ are summarized in the appendix.

*Proof.* From corollaries 3.3, 5.3 and the relation

$$\langle j(x)j(0)\rangle_T=\frac{1}{2}\frac{\partial^2}{\partial x^2}\langle Q(x)^2\rangle_T=\frac{1}{2}\frac{\partial^2}{\partial x^2}\ \frac{\partial^2}{\partial\alpha^2}\langle\exp\left(\alpha Q(x)\right)\rangle_T\bigg|_{\alpha=0} \qquad (5.27)$$

we can derive the result. For example the constant $D^2$ is derived by

$$D^2=\left(1+\frac{2}{c}D\right)\frac{1}{2}\frac{\partial^2}{\partial x^2}\frac{\partial^2}{\partial\alpha^2}\ A\left(\beta,\alpha-\frac{\sqrt{2mT}}{c}C(\beta,\alpha)\right)\mathrm{e}^{-C(\beta,\alpha)y}\bigg|_{\alpha=0}+\cdots\ . \qquad (5.28)$$

□

Korepin [8] proposed a method of presenting correlation functions in the form of special series. This method is useful in the calculation of the long-distance asymptotics. Bogoliubov and Korepin [9] considered the asymptotics of correlation functions for the penetrable Bose gas by the special series method. Corollary 5.3 coincides with the result of [9]. For the impenetrable Bose gas case ($c=\infty$), Korepin and Slavnov [10] calculated higher-order corrections and derived pre-exponential polynomials by the special series method. In this paper we derived pre-exponential polynomials for penetrable Bose gas case ($0<c<+\infty$) by using the determinant representation.

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## Appendix

In this appendix we summarize the asymptotics of the density-density correlation function. We use the notation given in corollary 5.3. In what follows, we use the following abbreviations:

$$\lambda_1=\lambda_1(0)=\sqrt{\beta+\pi\mathrm{i}}\qquad r_1=|\lambda_1|=|\lambda_1(0)|\qquad \varphi_1=\arg\lambda_1=\arg\lambda_1(0). \qquad (A.1)$$

First we summarize the coefficients of $\exp\{-4r_1\sin\varphi_1\sqrt{mT/c}\,x\}$:

$$\frac{mT}{2}\left(B_0(\beta)+B_1(\beta)\sqrt{\frac{mT}{2}}x+B_2(\beta)\left(\sqrt{\frac{mT}{2}}x\right)^2\right)\exp\left\{-4r_1\sin\varphi_1\sqrt{\frac{mT}{2}}x\right\}. \qquad (A.2)$$

The functions $B_j(\beta)$ are given by

$$B_j(\beta) = \tilde{B}_j(\beta)\left(1 + \frac{2}{c}D\right) \qquad (j = 0, 1, 2) \tag{A.3}$$

where the $\tilde{B}_j(\beta)$ are given by

$$\tilde{B}_2(\beta) = 16r_1^2 \sin^2\varphi_1 \left(\frac{d(\beta)}{\pi} + \frac{2\sin\varphi_1}{r_1}\right)^2 \times B\,(\beta, -4\Delta r_1 \sin\varphi_1) \tag{A.4}$$

$$\begin{aligned}
\tilde{B}_1(\beta) = {} & 32r_1^2 \sin^2\varphi_1 \left(\frac{d(\beta)}{\pi} + \frac{2\sin\varphi_1}{r_1}\right)\left(1 + \Delta\left(\frac{d(\beta)}{\pi} + \frac{2\sin\varphi_1}{r_1}\right)\right) \\
& \times \left(\frac{\partial B}{\partial\alpha}\right)(\beta, -4\Delta r_1 \sin\varphi_1) \\
& - 4r_1 \sin\varphi_1 \left\{\left(\frac{2d(\beta)}{\pi}\right)^2 + \frac{16\sin\varphi_1}{\pi r_1} d(\beta) - \frac{4r_1\sin\varphi_1}{\pi}\left(\frac{\partial d}{\partial\beta}\right)(\beta)\right. \\
& \left. + \frac{4\sin^2\varphi_1}{r_1^2}(5 + \cos 2\varphi_1)\right\} \times B(\beta, -4\Delta r_1 \sin\varphi_1)
\end{aligned} \tag{A.5}$$

$$\begin{aligned}
\tilde{B}_0(\beta) = {} & 16r_1^2 \sin^2\varphi_1 \left\{1 + \Delta\left(\frac{d(\beta)}{\pi} + \frac{2\sin\varphi_1}{r_1}\right)\right\}^2 \times \left(\frac{\partial^2 B}{\partial\alpha^2}\right)(\beta, -4\Delta r_1 \sin\varphi_1) \\
& - 4r_1 \sin\varphi_1 \left[\frac{4d(\beta)}{\pi} + \frac{8\sin\varphi_1}{r_1} + \Delta\left\{\left(\frac{2d(\beta)}{\pi}\right)^2 + \frac{16\sin\varphi_1}{\pi r_1} d(\beta)\right.\right. \\
& \left.\left. - \frac{4r_1\sin\varphi_1}{\pi}\left(\frac{\partial d}{\partial\beta}\right)(\beta) + \frac{4\sin^2\varphi_1}{r_1^2}(5 + \cos 2\varphi_1)\right\}\right] \\
& \times \left(\frac{\partial B}{\partial\alpha}\right)(\beta, -4\Delta r_1 \sin\varphi_1) \\
& + 2\left\{\left(\frac{d(\beta)}{\pi}\right)^2 + \frac{4\sin\varphi_1}{\pi r_1} d(\beta) - \frac{4r_1\sin\varphi_1}{\pi}\left(\frac{\partial d}{\partial\beta}\right)(\beta) + \frac{4\sin^2 2\varphi_1}{r_1^2}\right\} \\
& \times B\,(\beta, -4\Delta r_1 \sin\varphi_1)\,.
\end{aligned} \tag{A.6}$$

Here we used the abbreviations

$$\Delta = \frac{\sqrt{2mT}}{c} \qquad d(\beta) = \int_{-\infty}^{\infty} \frac{1}{1 + \mathrm{e}^{\mu^2 - \beta}}\,\mathrm{d}\mu \quad \beta = \frac{h}{T} \tag{A.7}$$

and the function $B(\beta, \alpha)$ as defined in (5.3).

Next we summarize the coefficients of $\exp\{4\mathrm{i}\lambda_1\sqrt{mT/2}\,x\}$:

$$\frac{mT}{2}\left(G_0(\beta) + G_1(\beta)\sqrt{\frac{mT}{2}}x + G_2(\beta)\left(\sqrt{\frac{mT}{2}}x\right)^2\right)\exp\left\{4\mathrm{i}\lambda_1\sqrt{\frac{mT}{2}}x\right\}. \tag{A.8}$$

The functions $G_j(\beta)$ are given by

$$G_j(\beta) = \tilde{G}_j(\beta)\left(1 + \frac{2}{c}D\right) \qquad (j = 0, 1, 2). \tag{A.9}$$

where $\tilde{G}_j(\beta)$ are given by

$$\tilde{G}_2(\beta) = -16\lambda_1^2\left(\frac{d(\beta)}{\pi} + \frac{2\mathrm{i}}{\lambda_1}\right)^2 \times G\,(\beta, 4\Delta i\lambda_1) \tag{A.10}$$

$$\begin{aligned}\tilde{G}_1(\beta) = {}& -32\lambda_1^2\left(\frac{d(\beta)}{\pi} + \frac{2\mathrm{i}}{\lambda_1}\right)\left(1 + \Delta\left(\frac{d(\beta)}{\pi} + \frac{2\mathrm{i}}{\lambda_1}\right)\right) \times \left(\frac{\partial G}{\partial\alpha}\right)(\beta, 4\Delta i\lambda_1)\\ & + 4\mathrm{i}\lambda_1\left\{\left(\frac{2d(\beta)}{\pi}\right)^2 + \frac{16\mathrm{i}}{\pi\lambda_1}d(\beta) + \frac{4\mathrm{i}\lambda_1}{\pi}\left(\frac{\partial d}{\partial\beta}\right)(\beta) - \frac{12}{\lambda_1^2}\right\}\\ & \times G(\beta, 4\Delta\mathrm{i}\lambda_1)\end{aligned} \tag{A.11}$$

$$\begin{aligned}\tilde{G}_0(\beta) = {}& -16\lambda_1^2\left\{1 + \Delta\left(\frac{d(\beta)}{\pi} + \frac{2\mathrm{i}}{\lambda_1}\right)\right\}^2 \times \left(\frac{\partial^2 G}{\partial\alpha^2}\right)(\beta, 4\Delta\mathrm{i}\lambda_1)\\ & + 4\mathrm{i}\lambda_1\left[\frac{4d(\beta)}{\pi} + \frac{8\mathrm{i}}{\lambda_1} + \Delta\left\{\left(\frac{2d(\beta)}{\pi}\right)^2 + \frac{16\mathrm{i}}{\pi\lambda_1}d(\beta)\right.\right.\\ & \left.\left. + \frac{4\mathrm{i}\lambda_1}{\pi}\left(\frac{\partial d}{\partial\beta}\right)(\beta) - \frac{12}{\lambda_1^2}\right\}\right] \times \left(\frac{\partial G}{\partial\alpha}\right)(\beta, 4\Delta\mathrm{i}\lambda_1)\\ & + 2\left\{\left(\frac{d(\beta)}{\pi}\right)^2 + \frac{4\mathrm{i}}{\pi\lambda_1}d(\beta) + \frac{4\mathrm{i}\lambda_1}{\pi}\left(\frac{\partial d}{\partial\beta}\right)(\beta)\right\} \times G\,(\beta, 4\Delta\mathrm{i}\lambda_1)\,.\end{aligned} \tag{A.12}$$

Here the function $G(\beta, \alpha)$ is defined in (5.4). Next we summarize the coefficients of $\exp\{-4\mathrm{i}\lambda_1^*\sqrt{mT/2}\,x\}$:

$$\frac{mT}{2}\left(H_0(\beta) + H_1(\beta)\sqrt{\frac{mT}{2}}x + H_2(\beta)\left(\sqrt{\frac{mT}{2}}x\right)^2\right)\exp\left\{-4i\lambda_1^*\sqrt{\frac{mT}{2}}x\right\} \tag{A.13}$$

where $\lambda_1^* = \sqrt{\beta - \pi\mathrm{i}}$. The functions $H_j(\beta)$ are given by the complex conjugation of $G_j(\beta)$.

$$H_j(\beta) = G_j^*(\beta) \qquad (j = 0, 1, 2). \tag{A.14}$$

## Beferences


[1] Korepin V E, Bogoliubov N M and Izergin A G 1993 *Quantum Inverse Scattering Method and Correlation Functions* (Cambridge Monographs on Mathematical Physics) (Cambridge: Cambridge University Press)

[2] Lieb E H and Liniger W 1963 Exact analysis of an interacting Bose gas I. The general solution and the ground state *Phys. Rev.* **130** 1605–16

[3] Lieb E H 1963 Exact analysis of an interacting Bose gas II. The excitation spectrum *Phys. Rev.* **130** 1616–34

[4] Yang C N and Yang C P 1969 Thermodynamics of a one dimensional systems of bosons with repulsive delta-function interactions *J. Math. Phys.* **10** 1122–52

[5] Burtsev S P, Gatitov I R and Zakharov V E 1988 Maxwell–Bloch system with pumping *Plasma Theory and Nonlinear and Turbulent Processes in Physics* vol 2 (Singapore: World Scientific) pp 897–905

[6] Korepin V E 1989 Generating functional of correlation functions for the nonlinear Schrödinger equation *Funct. Analiz. Prilozh.* **23** 15–23 (in Russian)

[7] Jimbo M, Miwa T, Môri Y and Sato M 1980 Density matrix of an impenetrable Bose gas and the fifth Painlevé transcendent *Physica* **1D** 80–158
[8] Korepin V E 1984 Correlation functions of the one dimensional Bose Gas in the repulsive case *Commun. Math. Phys.* **94** 93–113
[9] Bogoliubov N M and Korepin V E 1985 Correlation length of the one-dimensional Bose gas *Nucl. Phys.* B [FS14] 766–78
[10] Korepin V E and Slavnov N A 1986 Correlation function of currents in a one-dimensional Bose gas *Theor. Math. Phys.* **68** 955–60
[11] Frenkel I B and Jing N 1988 Vertex representations of quantum affine algebras *Proc. Natl Acad. Sci. USA* **85** 9373–7
[12] Essler F H, Frahm H, Its A R and Korepin V E 1996 Painlevé transcendent describes quantum correlation function of the $XXZ$ antiferromagnet away from the free-fermion point *Yukawa preprint* YITP-96-13, solv-int/9604005